

## Exact spherically symmetric solutions in massive gravity

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**ABSTRACT:** A phase of massive gravity free from pathologies can be obtained by coupling the metric to an additional spin-two field. We study the gravitational field produced by a static spherically symmetric body, by finding the exact solution that generalizes the Schwarzschild metric to the case of massive gravity. Besides the usual  $1/r$  term, the main effects of the new spin-two field are a shift of the total mass of the body and the presence of a new power-like term, with sizes determined by the mass and the shape (the radius) of the source. These modifications, being source dependent, give rise to a dynamical violation of the Strong Equivalence Principle. Depending on the details of the coupling of the new field, the power-like term may dominate at large distances or even in the ultraviolet. The effect persists also when the dynamics of the extra field is decoupled.

**KEYWORDS:** Classical Theories of Gravity, Black Holes, Space-Time Symmetries.

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## 1. Introduction

The search for large-distance modified theories of gravity, motivated by the evidence for the cosmological acceleration, has stimulated a number of studies in the recent years. The main goal has been to look for a massive deformation of standard general relativity, featuring a large distance (infrared) modification of the Newtonian gravitational potential, and massive gravitons.

The idea of considering a Lorentz-invariant theory of a massive spin-two field dates back to 1939 [1]: the resulting theory is plagued by a number of diseases that make it unphysical, besides being phenomenologically excluded. In particular, the modification of the Newtonian potentials is not continuous when the mass  $m^2$  vanishes, giving a large correction (25%) to the light deflection from the sun that is experimentally excluded [2]. A possible way to circumvent the physical consequences of the discontinuity was proposed in [3]; the idea is that in the Fierz Pauli theory (FP) the linearized approximation breaks

down near the star and an improved perturbative expansion must be used, leading to a continuous result when  $m \rightarrow 0$ . Whether the solution associated with the improved perturbative expansion valid near the star can be extended up to infinity is an open problem [4]. In addition, FP is problematic as an effective theory at the quantum level. Regarding FP as a gauge theory where the gauge symmetry is broken by an explicit mass term  $m$ , one would expect a cutoff  $\Lambda_2 \sim mg^{-1} = (mM_{pl})^{1/2}$ , however the real cutoff is  $\Lambda_5 = (m^4 M_{pl})^{1/5}$  [5] much lower than  $\Lambda_2$ . A would-be Goldstone mode is responsible for the extreme ultraviolet sensitivity of FP theory, that becomes totally unreliable in the absence of proper completion. These issues cast a shadow on the possibility of realizing a Lorentz-invariant theory of massive gravity [6].

It was recently noted that by allowing lorentz-breaking mass terms for the graviton the resulting theory can be physically viable [7], being free from pathologies such as ghosts or low strong coupling scales, while still leading to modified gravity. Since the mass terms break anyway the diffeomorphisms invariance, this possibility was analyzed mainly in a model-independent way, by reintroducing the goldstone fields of the broken gauge invariance, and by studying their dynamics [5, 8]; we refer to a recent review for the status and results in this direction [9]. This approach has the power of being model independent, but the advantage turns into a difficulty when investigating the concrete behavior of solutions.

In [10] we considered a class of theories that generate Lorentz-breaking mass terms for the graviton, by coupling the metric to an additional spin-two field. This system was originally introduced and analyzed by Isham, Salam and Strathdee [11, 12] and reanalyzed more recently in [13–15]. While this approach may seem antieconomical, we stress that this is the simplest model that can explain dynamically the emergence of lorentz breaking and give mass to the graviton. What happens is that the two tensor fields lead in general to two coexisting and different backgrounds, inducing Lorentz-breaking mass terms at linearized order. For a general discussion on the consequences of Lorentz Breaking see [16].

The linearized analysis showed that only two gravitons propagate, one massive and the other massless, both with two polarization states, representing two kinds of gravitational waves (GW). These are the only states in the theory that feel the Lorentz breaking, showing a frame-dependence that may be measured at future GW detectors.

In addition, the linearized gravitational potential differs in a crucial way from the Newtonian one: it contains a new term that is linearly growing with distance. Of course, this signals the breakdown of perturbation theory at large distances, and in this regime the theory should be treated fully nonlinearly. This fact is not surprising, since one has effectively introduced nonlinear interactions, and therefore antiscreening may be present also at classical level like in non-abelian gauge theories. We believe that this is a general feature of massive gravity theories due to the presence in the full theory of nonderivative interaction terms. In such situations the linearized analysis is of limited reach, and we are forced to find exact classical solutions to be compared with the standard Schwarzschild metric. Though in general this is a very hard task, we have managed to find a whole class of interaction terms for which nontrivial and rather interesting exact solutions can be found.

After describing the setup and the flat backgrounds in section 2 ad 2.1, we review the linearized analysis and its problems in section 2.2. We then describe the spherically

symmetric solutions in section 3, that we match with an interior star solution to estimate the modifications of the gravitational potential as a function of the source parameters. We also comment on the properties of these solutions and of the whole theory in the interesting limit when the second metric decouples, leaving just one massive gravity theory with modified Schwarzschild solutions, as well as in the Lorentz-invariant limit.

## 2. The model

Consider a gravity theory in which, besides our standard metric field, an additional rank-2 tensor is introduced in the form of a bimetric theory. The action is taken as<sup>1</sup>

$$S = \int d^4x \left[ \sqrt{-g_1} (M_{pl1}^2 R_1 + \mathcal{L}_1) + \sqrt{-g_2} (M_{pl2}^2 R_2 + \mathcal{L}_2) - 4(g_1 g_2)^{1/4} V(X) \right], \quad (2.1)$$

and for symmetry each rank-2 field is coupled to its own matter with the respective Lagrangians  $\mathcal{L}_{1,2}$ . In the interaction term we only consider non-derivative couplings. The only invariant tensor, without derivatives, that can be written out of the two metrics is  $X_\nu^\mu = g_1^{\mu\alpha} g_{2\alpha\nu}$ , and then  $V$  is taken as a function of the four independent scalars  $\{\tau_n = \text{tr}(X^n), n = 1, 2, 3, 4\}$  made out of  $X$ . The cosmological terms can be included in  $V$ , e.g.  $V_{\Lambda_1} = \Lambda_1 q^{-1/4}$ , with  $q = \det X = g_2/g_1$ .

Then the (modified) Einstein equations read

$$M_{pl1}^2 E_{1\nu}^\mu + Q_{1\nu}^\mu = \frac{1}{2} T_{1\nu}^\mu \quad (2.2)$$

$$M_{pl2}^2 E_{2\nu}^\mu + Q_{2\nu}^\mu = \frac{1}{2} T_{2\nu}^\mu, \quad (2.3)$$

where we defined the effective energy-momentum tensors induced by the interaction:

$$Q_{1\nu}^\mu = q^{1/4} [V \delta_\nu^\mu - 4(V' X)_\nu^\mu] \quad (2.4)$$

$$Q_{2\nu}^\mu = q^{-1/4} [V \delta_\nu^\mu + 4(V' X)_\nu^\mu], \quad (2.5)$$

with  $(V')_\nu^\mu = \partial V / \partial X_\mu^\nu$ . Indeed, the field  $g_2$  plays the role of matter in the equations of motion for  $g_1$ , and viceversa for  $g_1$ .

The Einstein tensors satisfy the corresponding contracted Bianchi identities<sup>2</sup>

$$g_1^{\alpha\nu} \nabla_{1\alpha} E_{1\mu\nu} = \nabla_1^\nu E_{1\mu\nu} = 0 \quad g_2^{\alpha\nu} \nabla_{2\alpha} E_{2\mu\nu} = \nabla_2^\nu E_{2\mu\nu} = 0. \quad (2.6)$$

that follows from the invariance of the respective Einstein-Hilbert terms under common diffeomorphisms

$$\delta g_{1\mu\nu} = 2g_{1\alpha(\mu} \nabla_{1\nu)} \xi^\alpha \quad \delta g_{2\mu\nu} = 2g_{2\alpha(\mu} \nabla_{2\nu)} \xi^\alpha. \quad (2.7)$$

The interaction term is also separately invariant and we can derive conservation laws for  $Q_1$  and  $Q_2$  similar to the conservation of the energy-momentum tensor in GR:

$$\begin{aligned} \nabla_1^\nu Q_{1\mu\nu} &= 0 && \text{on shell for } g_2 \\ \nabla_2^\nu Q_{2\mu\nu} &= 0 && \text{on shell for } g_1. \end{aligned} \quad (2.8)$$

<sup>1</sup>We use the mostly plus convention for the metric. Indices of type 1(2) are raised/lowered with  $g_1(g_2)$ .

<sup>2</sup> $\nabla_{1/2}$  denotes the covariant derivative associated to the Levi-Civita connection of  $g_{1/2}$ .

These identities are quite powerful; for instance they allow to solve completely the simplest of these models, when  $V$  is a function of  $q$  only. This peculiar case is discussed in appendix D.

### 2.1 Asymptotic solutions

At infinity, far from all the sources, we expect that  $g_1$  and  $g_2$  are maximally symmetric, setting up the benchmark for the asymptotic behavior of all solutions of the EOM. Denoting with  $-\mathcal{K}_a/4$  the constant scalar curvature of  $g_{a\mu\nu}$ , i.e.

$$M_{pl a}^2 E_{a\mu\nu} = \mathcal{K}_a g_{a\mu\nu}, \quad (a = 1, 2) \tag{2.9}$$

the equations (2.2)–(2.3) read

$$2V + \left( q^{-1/4} \mathcal{K}_1 + q^{1/4} \mathcal{K}_2 \right) = 0 \tag{2.10}$$

$$8(V'X)^\mu_\nu + \delta^\mu_\nu \left( q^{1/4} \mathcal{K}_2 - q^{-1/4} \mathcal{K}_1 \right) = 0. \tag{2.11}$$

and these equations can be solved for specific ansätze. In order to study the properties of this model for asymptotically flat spaces, we analyze first the biflat solutions,  $\mathcal{K}_1 = \mathcal{K}_2 = 0$ . Eqs. (2.10)–(2.11) yield:

$$V'^\nu_\mu = 0, \quad V = 0. \tag{2.12}$$

Assuming that rotational symmetry is preserved and that the two metrics have the same signature, the biflat solution can be written in the following form

$$\begin{aligned} \bar{g}_{1\mu\nu} &= \eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1) \\ \bar{g}_{2\mu\nu} &= \omega^2 \text{diag}(-c^2, 1, 1, 1), \end{aligned} \tag{2.13}$$

where  $c$  parametrizes the speed of light in sector 2 and  $\omega$  is the relative conformal factor. Eqs. (2.12) correspond to three independent equations  $V = 0$ ,  $V'^0_0 = 0$  and  $V'^i_i = 0$ , where 0 and  $i = 1, 2, 3$  stand for temporal and spatial indices. Therefore, two of these equations determine the values of two parameters  $c$  and  $\omega$ , while the third represents a fine tuning condition for the function  $V$ , necessary to ensure flatness. The same sort of fine tuning is necessary in the context of normal GR to set the cosmological term to zero. Therefore, for a generic function  $V(X)$  we expect to have a Lorentz Breaking (LB) solution with  $c \neq 1$ , and hence a preferred reference frame (2.13) in which both metrics are diagonal. Asymptotic flat solutions should approach (2.13) in a suitable coordinate system. In addition, there exists also a Lorentz Invariant (LI) solution, with  $c = 1$ : in this case two equations coincide  $V'^0_0 = V'^i_i$ , and can be used to determine the value of  $\omega$ .

Summarizing, asymptotically the solutions fall in two branches: LI with  $c = 1$ , and LB with  $c \neq 1$ .<sup>3</sup> The LB branch is of particular interest since it naturally allows for consistent massive deformations of gravity [10].

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<sup>3</sup>In the special case when  $V = V(\det X)$ , Bianchi identities force  $\det X$  to be constant, and there are additional gauge symmetries that allow to set  $c = 1$ . As a result the branches are equivalent (appendix D).

## 2.2 Review (and critics) of the linearized analysis

In [10] we performed a linearized analysis around the biflat background, that we report here for the LB branch.

In addition to the kinetic terms, the linearized action contains a Lorentz-breaking mass term for the fluctuations  $h_{a\mu\nu} = g_{a\mu\nu} - \bar{g}_{a\mu\nu}$  ( $a = 1, 2$ ). Since on the biflat background  $V = V'_{\mu} = 0$ , one can expand the potential at second order in the fluctuations  $Y = \bar{X}\bar{g}_2^{-1}h_2 - \bar{g}_1^{-1}h_1\bar{X}$ , and define the mass lagrangian:

$$\begin{aligned} \mathcal{L}_m &= -2(\bar{g}_1\bar{g}_2)^{1/4} \text{tr} [Y V''(\bar{X}) Y] \\ &\equiv \frac{1}{4} \left( h_{00}^t \mathcal{M}_0 h_{00} + 2h_{0i}^t \mathcal{M}_1 h_{0i} - h_{ij}^t \mathcal{M}_2 h_{ij} + h_{ii}^t \mathcal{M}_3 h_{ii} - 2h_{ii}^t \mathcal{M}_4 h_{00} \right). \end{aligned} \quad (2.14)$$

Here  $h_{\mu\nu} = \{h_{1\mu\nu}, h_{2\mu\nu}\}$  is the column vector of fluctuations and  $\mathcal{M}_{0,1,2,3,4}$  are  $2 \times 2$  mass matrices. It is then crucial to realize that, due to linearized gauge invariance (that we remark is never broken) these matrices are of rank-one; one can write

$$\begin{aligned} \mathcal{M}_0 &= \lambda_0 \mathcal{C}^{-2} \mathcal{P} \mathcal{C}^{-2} \\ \mathcal{M}_{2,3} &= \lambda_{2,3} \mathcal{P}, \quad \mathcal{P} \equiv \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \quad \mathcal{C} \equiv \begin{pmatrix} 1 & \\ & c \end{pmatrix} \\ \mathcal{M}_4 &= \lambda_4 \mathcal{C}^{-2} \mathcal{P} \end{aligned} \quad (2.15)$$

where  $\lambda_{0,2,3,4}$  depend on the potential. In addition, due to the LB,  $\mathcal{M}_1$  vanishes regardless of the potential  $V$ . This fact leads to a well defined phase of linearized massive gravity.<sup>4</sup>

In this phase, the only propagating states are the two spin-2 tensor components of the fluctuations (two polarizations each) corresponding to two gravitons, of which one is massless and the other has mass  $\lambda_2$ . Their dispersion relation is non-linear due to their mixing and their different propagating speeds [10].

The other components, scalars and vectors, do not propagate, and therefore discontinuity and strong coupling problems are absent in this phase. They however mediate instantaneous interactions, so the Newtonian potentials that one finds at linearized level are then drastically modified; for example in sector 1, the potential from a point-like source  $M_1$  is:

$$\Phi_1 = -\frac{GM_1}{r} + GM_1\mu^2 r, \quad (2.16)$$

where

$$\mu^2 \equiv \frac{\lambda_2}{2M_1^2} \frac{3\lambda_4^2 - \lambda_0(3\lambda_3 - \lambda_2)}{\lambda_4^2 - \lambda_0(\lambda_3 - \lambda_2)} \quad (2.17)$$

and for later reference note that  $\mu^2$  may be negative.

The linearly growing term in (2.16) signals the breakdown of perturbation theory at distances larger than  $r_{\text{IR}} = (GM_1\mu^2)^{-1}$  [8, 10], and one usually considers the solution to be valid as long as the potential stays in the weak field regime. However, one should note

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<sup>4</sup>The vanishing of the second eigenvalue of  $\mathcal{M}_1$  can be understood by noting that  $h_{10i} - h_{20i}$  is a goldstone direction, corresponding to the broken boosts in the LB background. In section 3.6 we comment on the fate of this condition on nontrivial backgrounds.

that the linear term in (2.16) is induced by an other scalar field having an instantaneous interaction and acting as a source for  $\Phi$  (see e.g. [23, 21]). It is then easy to realize that non-linear corrections to this field can drastically modify the IR behavior, even in the weak-field regime. We can clarify this point by showing, as an example, two systems of differential equations that differ by nonlinear terms and have drastically different IR behavior

$$\begin{cases} \Delta\Phi + \mu^2\sigma = M\delta^3(x) \\ \Delta\sigma = M\delta^3(x) \end{cases} \quad \begin{cases} \Delta\Phi + \mu^2\sigma = M\delta^3(x) \\ \Delta\sigma + \lambda\sigma^2 = M\delta^3(x), \end{cases} \quad (2.18)$$

Here  $\Phi$  is a scalar field (mimicking the gravitational potential)  $\sigma$  is an additional scalar field coupled to it by a mass term (and  $\Delta$  is the laplacian). While in the first system  $\sigma \sim M/r$  and this induces a linear term in  $\Phi$  like in (2.16), in the second system  $\sigma$  drops to zero faster than  $M/r$  so that the bad behavior of  $\Phi$  is cured. What happens is that the IR behavior is dominated by a non-linear term, because effectively  $\Delta \rightarrow 0$  at large distances. We incidentally point out that standard GR is safe in this respect, because nonlinear terms coming from the Einstein tensor are always accompanied by two derivatives and thus are equally suppressed at large distances.

We are thus led to the conclusion that in massive gravity the situation is similar to non-abelian gauge theories, where the large distance behavior is generically non-trivial and inaccessible to the linearized approximation. We recall that in Yang-Mills theories, non-abelian configurations of charges can lead to non coulomb-like classical solutions, screening or even anti-screening the charge, leading also to infinite energy configurations.<sup>5</sup>

In this situation what one may try is to really look at higher orders and maybe retain the first terms that are relevant at large distance. This approach would require the painful procedure of defining the gauge invariant fields at higher orders, and would also lead to nonlinear terms mixing scalars, vectors and tensors. Instead of following this approach, we find more instructive to study the exact spherically symmetric solutions.

### 3. Exact spherically symmetric static solutions

The Schwarzschild solution describes the spherically symmetric gravitational field produced by a spherically symmetric source. It is crucial to understand what kind of modification is introduced in this theory by the presence of a new spin 2 field. Spherical symmetry allows us to choose a coordinate patch  $(t, r, \theta, \varphi)$  where  $g_1$  and  $g_2$  have the form

$$ds_1^2 = -J dt^2 + K dr^2 + r^2 d\Omega^2 \quad (3.1)$$

$$ds_2^2 = -C dt^2 + A dr^2 + 2D dt dr + B d\Omega^2. \quad (3.2)$$

and all the functions  $J, K, C, A, D, B$  entering  $g_1$  and  $g_2$  are function of  $r$  only. Notice that the off-diagonal piece  $D$  cannot be gauged away.

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<sup>5</sup>Point-source solutions are usually discarded because they have infinite energy [17], but they can be retained if one allows the flux lines to end on other charges, see for example [18].

### 3.1 Black hole solutions

In the absence of matter, a number of interesting properties follow from the form of the Einstein tensors  $E_{1\nu}^\mu$ ,  $E_{2\nu}^\mu$  derived from (3.1)–(3.2) and do not depend on the chosen  $V$ . Following [12, 19] the spherically symmetric solutions can be divided in two classes: type I with  $D \neq 0$  and type II with  $D = 0$ . We shall focus here mainly on type I solutions.

Since  $E_{1\nu}^\mu$  is diagonal by the choice of the first metric, then also  $(V'X)_\nu^\mu$  must be diagonal because of the EOM (2.2). The only possible source of a off-diagonal term in the r.h.s. of (2.3) would be  $(V'X)_\nu^\mu$ , so as a result also  $E_{2\nu}^\mu$  must be diagonal, i.e.  $E_{2r}^t = 0$ . For type I solutions, this condition amounts to a single equation:

$$AC + D^2 = d_2 \frac{(B')^2}{4B}, \quad (3.3)$$

where  $d_2$  is a constant. Incidentally using this relation it turns out that  $E_{2t}^t = E_{2r}^r$ , then using (2.3) also  $(V'X)_t^t = (V'X)_r^r$ , and by (2.2) we have also that  $E_{1t}^t = E_{1r}^r$ . This relation determines  $K$  in terms of  $J$

$$K = \frac{d_1}{J}, \quad (3.4)$$

with  $d_1$  an other constant.

The metric 2 can be brought in a diagonal form by a coordinate change  $dt = dt' + dr D/C$ . Thanks to (3.3), in the new coordinates we have

$$ds_2^2 = -C dt'^2 + \frac{(B')^2 d_2}{4B C} dr^2 + B d\Omega^2 \quad (3.5)$$

(and of course the metric 1 in no longer diagonal). Then by a suitable change of  $r$  the metric 2 can also be put in a Schwarzschild-like form; setting  $r' = \sqrt{B(r)}$ , we find

$$ds_2^2 = -C dt'^2 + \frac{d_2}{C} dr'^2 + r'^2 d\Omega^2, \quad C(r) = C(r'), \quad (3.6)$$

which shows that  $C$  is the physically relevant potential in sector 2.

To proceed further a choice of  $V$  is needed. In the existing literature essentially all the results are based on a potential  $V_{\text{IS}}$  introduced in [20] and [11] in the context of hadronic physics.<sup>6</sup> The motivation for this choice is probably due to the fact that  $V_{\text{IS}}$  is the simplest potential producing a FP mass term in the (Lorentz-invariant) linearized limit:

$$\begin{aligned} V_{\text{IS}} &= (\tau_{-2} - \tau_{-1}^2 + 6\tau_{-1} - 12) \\ &= (g^{2\mu\nu} - g^{1\mu\nu})(g^{2\rho\sigma} - g^{1\rho\sigma})(g_{1\mu\rho}g_{1\nu\sigma} - g_{1\mu\nu}g_{1\rho\sigma}) \\ &\simeq \text{tr}(h_-^2) - \text{tr}(h_-)^2 \quad \text{for } \bar{g}_1 = \bar{g}_2 = \eta. \end{aligned} \quad (3.7)$$

where  $h_{-\mu\nu} = h_{2\mu\nu} - h_{1\mu\nu}$ . For  $V_{\text{IS}}$  it was shown in [12] that type-I solutions are always Schwarzschild-(A)dS, and it was recently realized that these solutions are present for any potential [21]. It turns out that  $V_{\text{IS}}$  can be deformed and there exists a whole family of potentials for which the exact spherically symmetric solutions can be found.

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<sup>6</sup>In the years preceding QCD, the proposal of Isham, Salam and Strathdee was that of a second metric mediating a strongly coupled interaction, responsible for confinement of quarks inside tiny black holes.



Let us consider the family of potentials

$$V = a_0 + a_1 V_1 + a_2 V_2 + a_3 V_3 + a_4 V_4 + b_1 V_{-1} + b_2 V_{-2} + b_3 V_{-3} + b_4 V_{-4} + q^{-1/4} \Lambda_1 + q^{1/4} \Lambda_2, \quad (3.8)$$

where we introduced the following combinations involving the generalized determinants (again  $\tau_n = \text{tr}(X^n)$  and  $\epsilon$  is the 4-index antisymmetric symbol)

$$\begin{aligned} V_0 &= \frac{1}{24|g_2|} (\epsilon \epsilon g_2 g_2 g_2 g_2) = 1 \equiv \frac{1}{24q} (\tau_1^4 - 6 \tau_2 \tau_1^2 + 8 \tau_1 \tau_3 + 3 \tau_2^2 - 6 \tau_4) \\ V_1 &= \frac{1}{6|g_2|} (\epsilon \epsilon g_2 g_2 g_2 g_1) = (\tau_{-1}) \equiv \frac{1}{6q} (\tau_1^3 - 3 \tau_2 \tau_1 + 2 \tau_3) \\ V_2 &= \frac{1}{2|g_2|} (\epsilon \epsilon g_2 g_2 g_1 g_1) = (\tau_{-1}^2 - \tau_{-2}) \equiv q^{-1} (\tau_1^2 - \tau_2) \\ V_3 &= \frac{1}{|g_2|} (\epsilon \epsilon g_2 g_1 g_1 g_1) = (\tau_{-1}^3 - 3 \tau_{-2} \tau_{-1} + 2 \tau_{-3}) \equiv 6 q^{-1} \tau_1 \\ V_4 &= \frac{1}{|g_2|} (\epsilon \epsilon g_1 g_1 g_1 g_1) = (\tau_{-1}^4 - 6 \tau_{-2} \tau_{-1}^2 + 8 \tau_{-1} \tau_{-3} + 3 \tau_{-2}^2 - 6 \tau_{-4}) \equiv 24 q^{-1} \end{aligned} \quad (3.9)$$

and where  $V_{-n} = V_n(X \rightarrow X^{-1})$ . The cosmological constants  $\Lambda_1$  and  $\Lambda_2$  have been added to simplify the asymptotic flatness conditions. The Isham-Storey potential is recovered by setting  $a_0 = -12$ ,  $a_1 = 6$ ,  $a_2 = 1$  and  $b_1 = b_2 = b_3 = b_4 = a_3 = a_4 = 0$ .

Remarkably, the general combination  $V$  of (3.8) leads to solvable equations for type I spherically symmetric solutions, and these can be found in a closed form (these equations are the main result of the paper):<sup>7</sup>

$$J = \left[ 1 - 2 \frac{G m_1}{r} + \mathcal{K}_1 r^2 \right] + 2G S r^\gamma, \quad KJ = 1, \quad (3.10)$$

$$C = c^2 \omega^2 \left[ 1 - 2 \frac{G m_2}{\kappa r} + \mathcal{K}_2 r^2 \right] - \frac{2G}{c \omega^2 \kappa} S r^\gamma, \quad D^2 + AC = c^2 \omega^4 \quad (3.11)$$

$$B = \omega^2 r^2, \quad A = \omega^2 \frac{\tilde{J} - \tilde{C} - \tilde{J} \tilde{S} r^{\gamma-2}}{\tilde{j}^2}, \quad (3.12)$$

with  $\{\tilde{J}, \tilde{C}\} = \{J, C\}/\omega^4(c^2 + 1)$ ,  $\tilde{S} = S/\lambda_2 [(c^2 - 1)(\gamma + 1)(\gamma - 2)/16\omega^2 c^{1/2}(c^2 + 1)]$ . The solution depends on the integration constants  $m_1, m_2$  and  $S$  and we have introduced  $G = 1/16\pi M_{pl1}^2$  and  $\kappa = M_{pl2}^2/M_{pl1}^2$ . The values of  $c^2$ ,  $\gamma$  and of the graviton mass  $\lambda_2/M_{pl1}^2$  of the linearized analysis (2.15) are given in terms of the coupling constants:

$$c^2 = -\frac{\tilde{a}_1 + 4\tilde{a}_2 + 6\tilde{a}_3}{\tilde{b}_1 + 4\tilde{b}_2 + 6\tilde{b}_3}, \quad \gamma = -\frac{4[(\tilde{a}_2 + 3\tilde{a}_3) - c^2(\tilde{b}_2 + 3\tilde{b}_3)]}{c^2(\tilde{b}_1 + 4\tilde{b}_2 + 6\tilde{b}_3)}, \quad \lambda_2 = \frac{2(\gamma - 2)}{\gamma} (\alpha_2 + 3\alpha_3), \quad (3.13)$$

where  $\tilde{a}_n = \omega^{-2n} a_n$ ,  $\tilde{b}_n = \omega^{2n} b_n$  and  $\alpha_n = (\tilde{a}_n - c^2 \tilde{b}_n)(c^2 - 1)/c^2$ . Notice that one may trade  $\tilde{a}_1$ ,  $\tilde{b}_1$  and e.g.  $\tilde{a}_2$  for, respectively,  $c^2$ ,  $\gamma$  and the graviton mass  $\lambda_2$ , showing that these may take any value for this class of potentials. When  $\gamma < 2$ , the  $\mathcal{K}_i$  are proportional to

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<sup>7</sup>This solvability is linked to the fact that the combinations  $V_n$  are actually the coefficients of the secular equation of  $X$  and are (multi)linear combinations of its eigenvalues  $\lambda_i$ :  $V_n = \sum_{i_1 > i_2 > \dots > i_n} \lambda_{i_1} \lambda_{i_2} \dots \lambda_{i_n}$ .

the constant asymptotic curvatures of  $g_i$ ; the explicit expressions are given in appendix A. Finally,  $\omega^2$  is also in general a free parameter that determines for example  $\Lambda_2$  to have  $\mathcal{K}_2 = 0$ , after having fine tuned  $\Lambda_1$  to set  $\mathcal{K}_1 = 0$ .

The expression (3.10) resembles the Schwarzschild-dS(AdS) solution but with a crucial difference: a  $r^\gamma$  term of magnitude  $S$  is present and it may alter significantly the behavior of the gravitational field, depending on whether  $\gamma < -1$ ,  $-1 < \gamma < 0$  or  $\gamma > 0$ .

Before discussing these solutions, let us comment on the Isham-Storey potential  $V_{IS}$  used traditionally. Since in this case all  $b_n$  vanish, it leads to a singular situation where  $c^2 \rightarrow \infty$  unless an additional fine tuning  $\omega^2 = 2/3$  is performed. Even choosing this case, the linearized analysis is ill defined due to an enhanced gauge invariance (see [10] for the case of  $\lambda_\eta = 0$ ), and moreover from (3.13) one has  $\gamma \rightarrow \infty$ . This is the reason why only standard Schwarzschild-(A)dS solutions were found.

Now, in order to shed light on the physical meaning of the various constants in the solution let us also compute the total gravitational energy, as measured with respect to backgrounds 1 or 2. In the stationary case this is the Komar energy that can be calculated as a surface integral on a sphere of large radius  $r_{\text{outer}} \rightarrow \infty$  (see appendix C). We find, for the two fields:

$$\begin{aligned} \mathcal{E}_1 &= m_1 + S \gamma r_{\text{outer}}^{\gamma+1} \\ \mathcal{E}_2 &= m_2 - \frac{c}{\omega^2} S \gamma r_{\text{outer}}^{\gamma+1}. \end{aligned} \tag{3.14}$$

From these expressions we see that only IR modifications with  $\gamma < -1$  will lead to finite total energy.

**Case  $\gamma < -1$ .** At very large distance the solution reduce to a maximally symmetric solution parametrized by  $\mathcal{K}_i$ . In particular one can set  $\mathcal{K}_1 = \mathcal{K}_2 = 0$  with a single fine tuning, determining the asymptotic conformal factor  $\omega^2$  as discussed above, so that the solution describes asymptotically flat metrics. Clearly because  $\gamma < -1$ , at large distances gravity is Newtonian, while at short/intermediate distances, depending on  $S$ , the presence of the additional spin 2 field has changed the nature of the gravitational force.

Since the large distance behavior is Newtonian, the total energy is finite; taking  $r_{\text{outer}} \rightarrow \infty$ , we find  $\mathcal{E}_1 = m_1$ ,  $\mathcal{E}_2 = m_2$ . In black hole solutions like these,  $m_1$  and  $m_2$  are just parameters, that can be related to the mass of a material object only when the solution is considered as the outer part of, for instance, a star. In the case of standard GR, for a star of radius  $R$  and mass density  $\rho$ , the total gravitational energy  $E$  is the total mass  $M = 4\pi R^3 \rho/3$ . Here, the interaction with  $g_2$  is turned on and we expect a contribution to this energy given by the interaction term  $Q_1$ . Its size should be controlled by  $V$  and by the matter itself, because this interaction energy also is turned on by the source. Moreover, by dimensional analysis the coefficient  $S$  of the  $r^\gamma$  term should also be a function of the *size* of the object, and not only on its mass. This can be understood intuitively as the failure of the Gauss theorem due to the presence of  $Q$  in the EOMs and of the  $r^\gamma$  term in the solution. Accordingly, the separate contribution of  $Q_1$  to the energy is not expressible as a flux on a 2-surface at infinity, as it happens for the total Komar energy. The explicit computation of this interaction energy for a star will be performed in section 3.3.

**Case  $\gamma > -1$ .** For simplicity, also in this case we set  $\mathcal{K}_1 = \mathcal{K}_2 = 0$  as discussed above, but note that because the new term induces a curvature  $R \sim r^{\gamma-2}$ , only when  $\gamma < 2$  we have that  $g_1$  and  $g_2$  are asymptotically flat and  $\omega^2$  can be interpreted as an asymptotical conformal factor. For these choices of  $\gamma$ , we have a solution such that  $Q_1$  does not vanish rapidly as  $r \rightarrow \infty$ , and compensates a slow fall-off (or rise!) of the gravitational field. However on dimensional grounds any fall-off slower than  $1/r$  makes the Komar total energy infinite, and indeed when  $r_{\text{outer}} \rightarrow \infty$  both  $\mathcal{E}_1$  and  $\mathcal{E}_2$  diverge making this configuration physically unfeasible.

If spherically symmetric solutions of infinite energy are surely not physical, this may only suggest that solutions will not be spherically symmetric, as it happens in non-abelian gauge theories. For example one may speculate that finite energy configurations will arrange in flux tubes of gravitation at large distance, between sources of type 1 and 2, as suggested by the different signs of  $S$  in  $J$  and  $C$ . In a similar 'confinement-like' scenario, the term  $r^\gamma$  may be screened dynamically by the self-arrangement of configurations of matter 1 and 2, so that effectively  $S \rightarrow 0$  at large distances, as suggested by the full star solutions that we will describe later.

To summarize, we found that exact black-hole solutions are modified in the IR or in the UV depending on the choice of the potential, and that this behavior is not captured by the linearized approximation. There are even cases where the behavior of the potential is not modified at all with respect to GR (e.g.  $\gamma = 2$  or  $\gamma = -1$ ) while the linearized approximation still shows a linear term.

It is also interesting to observe that in the limit in which the second metric decouples,  $M_{pl2} \rightarrow \infty$  (i.e.  $\kappa \rightarrow \infty$ ), and assuming that  $m_2$  and  $S$  remains finite, from the solution (3.10) we find that the term  $S r^\gamma$  remains in  $g_1$  while  $g_2$  becomes exactly flat. We will discuss the decoupling limit as well as the limit  $c^2 \rightarrow 1$  in section 3.4 and 3.5.

### 3.2 Comparison with the linearized solution

It is interesting to comment on the perturbative origin of the exact solutions. This can be addressed by looking at the asymptotical weak-field limit of the solution (for  $\gamma < -1$ ):

$$J, K, C, A \sim \text{const} + O(1/r) \quad D \sim O(\sqrt{1/r}). \quad (3.15)$$

The crucial observation is that  $D$  vanishes more slowly (and non-analytically) than the other components of the perturbations. As a result, this solution is not captured by the standard linearization, where all the perturbations have the same large distance fall-off.

Technically, the origin of this behavior can be traced back to the equation  $(Q_{1,2})_r^t = 0$ , that is algebraic:

$$D(r) \left[ \frac{A(r)C(r) + D(r)^2}{J(r)K(r)} + \frac{a_1 + 4a_2r^2B(r)^{-1} + 6a_3r^4B(r)^{-2}}{b_1 + 4b_2r^{-2}B(r) + 6b_3r^{-4}B(r)^2} \right] = 0 \quad (3.16)$$

This equation can be solved either with  $D = 0$  (type-II solutions) or with  $D \neq 0$  (type-I). In this last case, for the exterior solution, since it turns out that  $B = \omega^2 r^2$ , asymptotically the equation turns into the definition of the speed of light of  $g_2$  (as in (3.13)), while the deviations give the mentioned behavior of  $D \sim 1/\sqrt{r}$ .

From equation (3.16) we can also understand that standard linearization (around the LB background) can not distinguish between type-I and type-II at leading order: considering the standard perturbative expansion (with parameter  $\epsilon$ ) where  $D \sim \epsilon$ , this equation starts from order  $\epsilon^2$ . Also, at first order  $D$  can be gauged away: it does not appear neither in the linearized Einstein tensors (due to separate gauge invariances) nor in the mass terms (due to  $\mathcal{M}_1 = 0$ ). At higher orders however one must choose  $D = 0$ , otherwise there is a constraint on the fields  $A, C, J, K, B$  that are already determined at previous-orders. We reach the conclusion that the standard perturbation theory around the LB background may only approximate the solutions in the type-II branch (if any exist: we recall that nontrivial type-II solutions are not known).

On the other hand, it is interesting that the  $r^\gamma$  term can be recovered in a semi-linearized approach, where one solves exactly equation (3.16) and treats the remaining ones perturbatively. This will be done for the interior star solution and the result containing the  $r^\gamma$  terms can be found in appendix B, e.g. equation B.2. Alternatively, if one insists in solving perturbatively all the equations, the correct result can also be recovered by assuming  $D \sim \sqrt{\epsilon}$  while all the other fluctuations are still of order  $\epsilon$ , and retaining the first nonvanishing order.<sup>8</sup>

Exactly as in the comparison between the Newtonian and Schwarzschild solutions, the final difference between this semi-linearized and the exact solution is just that  $K = 1 + 2\Phi$  instead of  $K = J^{-1} = 1/(1 - 2\Phi)$  (and similarly for  $g_2$ ).

### 3.3 Interior solution

In order to determine the integration constants  $m_1, m_2$  and  $S$  one can imagine that (3.10) is the exterior portion of the solution describing a spherically symmetric star. We aim at finding the interior solution and then determine  $m_1, m_2, S$  by matching with the exterior one.

It is instructive to consider first in full generality a spherical star made of fluids of type 1 and 2, extending from the origin to radii  $R_1, R_2$ , stationary with respect to the respective metrics:

$$T_{1\mu}^\nu = \begin{pmatrix} -\rho_1 & & & \\ & p_1 & & \\ & & p_1 & \\ & & & p_1 \end{pmatrix}, \quad T_{2\mu}^\nu = \begin{pmatrix} -\rho_2 \frac{D}{C}(p_2 + \rho_2) & & & \\ 0 & p_2 & & \\ & & p_2 & \\ & & & p_2 \end{pmatrix}. \quad (3.17)$$

Like in the vacuum, since  $g_1, T_1$  and  $E_1$  are diagonal, so should be  $Q_{1/2}$ , i.e.  $Q_{1r}^t = 0$ . This equation, being the same as in the vacuum case, is exactly solvable for the class of potentials (3.8). The remaining equations are more involved in the presence of matter, but in linearized approximation the solution can be found analytically.

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<sup>8</sup>A similar approach was envisaged in [6] to find an asymptotically flat modified Schwarzschild solution for LI massive gravity, valid in the  $m^2 \rightarrow 0$  limit. Also in that case a field that is not determined at linearized level for  $m = 0$ , is found to vanish non-analytically as  $\sqrt{1/r}$ . However that solution is not valid beyond some distance scale, and there is probably no global extension [6]. In the present work, it is remarkable that the semi-linearized solution is also extendable to the exact one.

According to the discussion of section 3.2 this partial linearization corresponds to choosing the type-I class ( $D \neq 0$ ) also for the interior solution.

For simplicity, we consider a star made of an incompressible fluid of constant density and small pressure,  $p \ll \rho$ . The interior solution is then matched by requiring continuity of  $C$ ,  $J$ ,  $B$ ,  $K$  and of the derivatives  $C'$ ,  $B'$ . This procedure, in the physical case when  $\gamma < -1$ , determines exactly the exterior constants  $m_1$ ,  $m_2$ ,  $S$ .<sup>9</sup>

While the detailed solution is given in appendix B, we present here the instructive case  $R_1 = R_2 = R$ ,  $M_1, M_2 \neq 0$ , and then discuss the phenomenologically interesting case of only matter 1,  $M_2 = 0$ .

**Case with both kinds of matter.** For  $R_1 = R_2 = R$ , and setting  $M_{1,2} = 4\pi\rho_{1,2}R_{1,2}^3/3$ , the matching condition gives:

$$\begin{aligned} m_1 &= M_1 + \Delta M, & \Delta M &= \alpha \mu^2 R^2 \left( M_1 - \frac{\omega^2}{\kappa} M_2 \right) \\ m_2 &= M_2 - \Delta M / c \kappa \omega^2 \\ S &= \Delta M R^{-(\gamma+1)} 15 / (2\gamma - 1)(\gamma - 4), \end{aligned} \tag{3.18}$$

where  $\alpha = 8c^{1/2}\omega^2/5(\gamma + 1)(\gamma - 2)$  and  $\mu^2$  is the same constant that appears in the linearly growing potential (2.16) of the linearized analysis. In the exterior solution (3.10)  $S$  and  $\Delta M$  modify the form of standard Schwarzschild solution. The first modification is in the Newtonian terms, and amounts to a *mass shift* with respect to the standard values  $m_{1,2} = M_{1,2}$ . The second is the *new term*  $r^\gamma$ . Both are proportional to the same combination  $\Delta M$ .<sup>10</sup>

The mass shift  $\Delta M$  can be understood as the contribution of the interaction terms  $Q$  to the total energy, i.e. to the total mass as measured by the Newton law at large distance. To clarify this, it is useful to recall that in standard GR the Komar energy, written as a spatial volume integral  $(8\pi)^{-1} \int R \xi dv$  (see appendix C) can be rewritten as a volume integral of the matter energy-momentum tensor by means of the Einstein equations:  $\mathcal{E} = \mathcal{E}_T = (8\pi)^{-1} \int (2T_\mu^\nu - T \delta_\mu^\nu) \xi^\mu dv_\nu$ . This result is modified in massive gravity because the Einstein equations contain the additional energy-momentum tensor of interactions  $Q$ , and the additional contribution can be evaluated with its volume integral  $\mathcal{E}_Q = (8\pi)^{-1} \int (2Q_\mu^\nu - Q \delta_\mu^\nu) \xi^\mu dv_\nu$ . We remark that while the sum of these two integrals, being the total energy, can be expressed as a surface integral at infinity (see appendix C), they separately can not, and they can only be evaluated using the smooth interior and exterior solution. The result of the volume integral is (again finite only for  $\gamma < -1$ ):

$$\mathcal{E}_{T_{1,2}} = M_{1,2}, \quad \mathcal{E}_{Q_1} = -\kappa \mathcal{E}_{Q_2} = \Delta M. \tag{3.19}$$

This confirms that the mass shift  $\Delta M$  is a screening effect, due to the energy of the interacting fields in  $Q$ , and corresponding to the nonzero Ricci curvature even outside the source (see e.g. [22]).

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<sup>9</sup>In the unphysical case  $\gamma > -1$ , although the potentials  $J$  and  $C$  are regular, the field  $B$  develops a singularity  $r^{\gamma+1}$  in the origin, calling probably for a fully nonlinear interior solution.

<sup>10</sup>But see appendix B for the full case  $R_1 \neq R_2$ .

As a side remark, looking at the matching (3.18), we observe that we did *not* linearize in  $V$ , but neglecting terms higher order in the matter density one has effectively neglected higher orders in  $V \sim \mu^2$ . Indeed, the result depends on the two dimensionless parameters  $R^2\mu^2$  and  $GM/R$ , but at first order in  $GM/R$  only the first order in  $R^2\mu^2$  appears, i.e.  $GMR\mu^2$ , and the final result is smooth when the interaction vanishes,  $V \sim \mu^2 \rightarrow 0$ . This is opposed to the singular massless limit of Lorentz-Invariant (Fierz-Pauli) massive gravity.

**Case of normal matter.** Turning off  $M_2 = 0$ , we can focus on sector 1 and discuss more phenomenologically how normal gravity is modified by the presence of the additional spin two field.

From the matching condition we have, with  $M_1 = M$ :

$$\begin{aligned} m_1 &= M(1 + \alpha \mu^2 R^2), & m_2 &= -\alpha \mu^2 R^2 M / c \kappa \omega^2 \\ S &= \mu^2 M R^{1-\gamma} \quad 15\alpha / (2\gamma - 1)(\gamma - 4) \end{aligned} \tag{3.20}$$

and from (3.10) we find for the modified potential (ignoring the numerical factors):

$$\Phi \sim G M \left[ \frac{1}{r} (1 + \mu^2 R^2) + \mu^2 R \left( \frac{r}{R} \right)^\gamma \right]. \tag{3.21}$$

The mass shift is now equivalent to a rescaling of the Newton constant  $G(1 + \alpha\mu^2 R^2)$ , that depends on the source radius!!

We observe that for the sun<sup>11</sup> we have  $\mu^2 R^2 \sim 10^{-10}$ , assuming all coupling constants to be of the same order so that  $\mu \sim m_g \lesssim (10^{-20} \text{ eV}) \sim (100\text{AU})^{-1}$  (this limit corresponding to the rough experimental bound on the graviton mass from pulsar GW emission [8, 23, 10]). We thus see that for the sun the size dependence is negligible and unobservable, and even more so for the planets. The effect becomes important for objects of size  $\mu^{-1}$ . For instance, for large objects with  $R \gtrsim 10^5 R_{\text{sun}}$  (red giants, large gas clouds, galaxies...) the effect may be of order one, and induces a macroscopic modification of the Newton constant. For low density objects that we consider here, this modification does not depend on the mass but just on the object size; therefore, given the mass, a large sphere of gas has a larger effective newton constant. In the limit  $\mu^2 R^2 > 1$ , the surface potential would even scale as  $R^4$ , instead of  $R^2$  as in standard gravity. Moreover, remembering that  $\mu^2$  may be negative, the negative interaction energy could cause large fluids to antigravitate, hinting toward the acceleration of the cosmological solutions.

Then, the new term in the potential is of the form

$$\delta\Phi \sim G M \mu^2 R \left( \frac{r}{R} \right)^\gamma, \tag{3.22}$$

replacing the linear term  $G M \mu^2 r$  of the linearized analysis. The Newtonian and the new term will be competing at a critical distance  $r_c$  that also depends on  $\gamma$ :

$$r_c = R \left| \frac{\mu^2 R^2}{1 + \mu^2 R^2} \right|^{-\frac{1}{\gamma+1}}. \tag{3.23}$$

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<sup>11</sup> $R_\odot \simeq 5 \cdot 10^5 \text{ Km} = 5 \cdot 10^{15} \text{ eV}^{-1}$  and  $M_\odot = 10^{66} \text{ eV}$ .

Of course since  $\gamma < -1$  the relevant modification is ultraviolet, and is evident for  $r < r_c$  (while for  $\gamma > -1$  it would be infrared, for  $r > r_c$ ). To estimate  $r_c$ , we observe that since the exponent  $-1/(\gamma + 1)$  is positive, one always has  $r_c < R$  for  $\mu^2 > 0$ , and the critical distance is inside the star. This does not mean that there will be no observable effects, since even subleading modifications to the Newton potential may be measured (for example modifications of the gravitational potential of relative magnitude  $10^{-3/-5}$  are at the level the current solar-system tests). On the other hand for negative  $\mu^2$ , and in particular for  $\mu^2 R^2 < -1/2$ , one has  $r_c > R$  so that in a UV region near the source the gravitational potential has stronger fall-off. For  $\mu^2 R^2 \simeq -1$  we even find that  $r_c$  becomes infinite, so that the region of UV modification expands to larger and larger distances!

We can summarize the results in the physical phase  $\gamma < -1$ :

- For sources of dimension  $R < \mu^{-1}$ , the effects are: a mass shift equivalent to a small Newton-constant renormalization  $(1 + \mu^2 R^2)$ , and a subleading correction to the Newtonian potential,  $\delta\Phi \lesssim (r/R)^\gamma$ .
- For large sources, of dimension  $R > \mu^{-1}$ , the mass shift is more pronounced, and for negative  $\mu^2$  even the new  $r^\gamma$  term can become dominant in a region near the source.

As we see, even discarding the nonphysical and possibly confining branch  $\gamma > -1$ , the phenomenology of these modified static solutions appears to be quite rich, and deserves a separate analysis to confront their features with real physical systems, e.g. modified galactic gravitational field, gravitation of large sources, post-Newtonian analysis.

### 3.4 Decoupling the second metric

The idea of introducing a second metric and considering its decoupling limit, to have a second background at hand while disposing of its fluctuations, is not new and was indeed considered to tackle the problem of the nonlinear continuation of the FP massive gravity [4]. However, due to the singular Isham-Storey potential, or due to the ill-defined nature of the Lorentz-Invariant theory, this did not lead to significant advance. In this work we found some nonperturbative solutions of the full system, so we are in a position to control the decoupling limit  $M_{pl2} \rightarrow \infty$  ( $\kappa \rightarrow \infty$ ) in which the second gravity is effectively switched off.

First, as far as the propagating states are concerned, we recall from the linearized analysis [10] that in the flat background out of two gravitons only the first graviton survives the decoupling limit: it is massive (with two polarization states and mass  $G\lambda_2$ ) and has a normal dispersion relation.

For the nontrivial solutions, as anticipated, one may take this limit in the exterior solutions, once one checks that  $m_1$ ,  $m_2$  and  $S$  stay finite. This is indeed shown by the interior solution (3.18), therefore we directly find the result:

$$\Phi_1 = \frac{GM_1}{r}(1 + R^2\mu^2\alpha) + GM_1\mu^2R\left(\frac{r}{R}\right)^\gamma [15\alpha/(2\gamma - 1)(\gamma - 4)] , \quad (3.24)$$

$$\Phi_2 = 0. \quad (3.25)$$

Both the mass shift and the new term remain, but the second gravity disappeared: here the other metric is flat!

A look at the exact solution (3.10) in the decoupling limit shows that the limiting metric 2 is still nondiagonal ( $D \neq 0$ ). This means that  $g_2$  is only *gauge-equivalent* to  $\eta_2 = \omega^2 \text{diag}\{-c^2, 1, 1, 1\}$ , and that to make contact with this traditional minkowski diagonal vacuum one has to choose the gauge (3.5), where  $\bar{g}_1$  is not diagonal. Explicitly we find:

$$ds_1^2 = -\bar{J} dt^2 + 2\bar{D} dt dr + \bar{K} dr^2 + r^2 d\Omega^2, \quad ds_2^2 = \omega^2(-c^2 dt^2 + dr^2 + r^2 d\Omega^2), \quad (3.26)$$

with  $\bar{J} = J$ ,  $\bar{K} = J^{-1}(1 - \bar{D}^2)$ ,  $\bar{D} = -(c\omega)^{-1}J\sqrt{\omega^2 - A}$ . Notice that  $\bar{D}$  is defined by the deviation of  $A$  from  $\omega^2$ , and that still  $\bar{J}\bar{K} + \bar{D}^2 = 1$ .

We therefore note that, to recover the present solutions in effective massive gravity theories, where only  $g_1$  is dynamical and the Lorentz breaking is an external diagonal metric, one should look for nondiagonal configurations.

We also remark that while taking the decoupling limit has left us with a flat auxiliary metric, still there is curvature for metric 1 in the vacuum outside the sources, due to  $Q_1$  and  $Q_2$  being nonzero there, because of the  $r^\gamma$  term. Therefore the order of the limit matters, and one would not get the correct result if one were to assume a flat second metric *before* taking the decoupling limit. In other words, setting  $g_2$  to be flat in advance:  $g_2 = \bar{g}_2$ , we have in vacuum

$$\begin{aligned} E_{2\nu}^\mu &= 0 = V\delta_\nu^\mu + 4(V'X)_\nu^\mu \quad \text{so that} & (3.27) \\ M_{pl1}^2 E_{1\nu}^\mu &= 2V\delta_\nu^\mu \quad \longrightarrow V = \text{const by Bianchi} \longrightarrow (\text{Anti})\text{deSitter}. \end{aligned}$$

Instead, in the limit  $M_{pl2} \rightarrow \infty$  we have still  $g_{2\mu\nu} \rightarrow \bar{g}_{2\mu\nu}$ , but different solutions for  $g_1$ :

$$\begin{aligned} E_{2\nu}^\mu &\rightarrow 0 \quad \text{but} \quad V\delta_\nu^\mu + 4(V'X)_\nu^\mu \neq 0, \quad \text{and then} & (3.28) \\ M_{pl1}^2 E_{2\nu}^\mu &= (V\delta_\nu^\mu - 4V'_\nu{}^\mu) \neq \text{const} \quad \longrightarrow \text{non-trivial solutions}. \end{aligned}$$

Summarizing, the decoupling limit shows that the theory remains well behaved, consisting of a modified gravity with massive gravitons, while the auxiliary metric is flat and decoupled.

### 3.5 Lorentz-Invariant limit

The Lorentz Invariant limit  $c^2 \rightarrow 1$  is also interesting to address the Vainshtein's claim that nonlinear corrections actually cure the discontinuity problem in Pauli-Fierz theory [6]. Indeed, we find that the limit  $c^2 \rightarrow 1$  is well behaved, and the solutions retain their validity. In this limiting phase therefore, gravity is modified, but lorentz breaking disappears.

The linearized mass term is accordingly of the form  $a h_{\mu\nu}^2 + b h^2$ , however the limiting theory reached in this way is *not* the Fierz-Pauli one, where  $a + b = 0$  (and for this reason FP is free from coupled ghosts). Here, we get to a theory where  $a = 0$ ; in fact, since we approach the LI phase from the  $\lambda_1 = 0$  branch, and because in the LI limit one has  $\lambda_1 = \lambda_2 = a$ , we see that also the graviton mass vanishes in this limit, as can be checked with the expression (3.13). It is nevertheless worth to point out that also  $a = 0$  is a ghost-free theory like  $a + b = 0$ . This case is not usually considered because at the linearized level there is no massive graviton as a consequence of an additional gauge symmetry (three



transverse diffeomorphisms), see [24] and also the PF0 phase in [10]. Accordingly, no strong coupling problems are expected and no Vainshtein issues. This matches nicely with our model having good properties along all the LB branch, that survive also in the LI limit.

### 3.6 Local Lorentz breaking in a nontrivial background

While the asymptotic biflat metrics are Lorentz breaking, one may ask about the situation at finite distance. This will have definite interest when addressing the nonpropagation of ghosts in the described nontrivial background. In fact, we recall (section 2.2) that on flat background this is a consequence of  $\mathcal{M}_1 = 0$ , and this follows from gauge invariance together with the fact that locally boosts are spontaneously broken. Now, even in nontrivial background a spontaneous breaking of Lorentz will lead to flat directions of the potential. Whether this fact will be enough to lead to absence of ghosts and to stable configurations is under scrutiny, and goes beyond the scope of the present work.

To describe the local breaking of Lorentz at any given point in the nontrivial background, one chooses a local Lorentz frame ( $g_1 = \eta$ ) and simultaneously diagonalizes  $g_2$ . The Lorentz breaking is given by the entries of  $g_2$ , that are actually the eigenvalues of  $X$ . These are easily calculated (in polar coordinates  $t, r, \theta, \phi$ ):

$$\hat{X} = \omega^2 \{c^2 \xi^{-1}, \xi, 1, 1\}, \tag{3.29}$$

with  $\xi = \frac{1}{2} \left[ c^2 f_- + f_+ + \sqrt{(c^2 - 1)(c^2 f_-^2 - f_+^2)} \right], \quad f_{\pm} = 1 \pm \tilde{S} r^{\gamma-2}.$

Quite remarkably, that they do not depend on the masses  $m_{1,2}$  of the newtonian terms, and this is due to the nondiagonal structure given by  $D$ . One can easily check that for  $r \rightarrow \infty$  we have  $\hat{X} = \omega^2 \{c^2, 1, 1, 1\}$ , reproducing the asymptotical lorentz breaking ( $\gamma < -1$ ).

Then we see that in the case  $S = 0$  the eigenvalues are constant, so that at any distance the Lorentz breaking is the same:  $\hat{X} = \omega^2 \{c^2, 1, 1, 1\}$ . We have two pure Schwarzschild solutions in a configuration that at any point breaks local boosts but preserves rotations (these are the solutions found in [24]).

On the other hand, since in general  $S \neq 0$ , a star solution will break not only boosts but also local rotations, because the  $rr$  term is different from the  $\theta\theta$  and  $\phi\phi$  ones. For example at large but finite distance, where  $Sr^{\gamma-2}$  is small, we have:

$$\hat{X} \simeq \omega^2 \{c^2(1 - \tilde{S}r^{\gamma-2}), 1 + \tilde{S}r^{\gamma-2}, 1, 1\}. \tag{3.30}$$

To compare the situation with standard GR, we recall that in GR Lorentz-invariance at any given point is always valid, in the Lorentz frame, and is broken in a finite neighbourhood only by the curvature (tidal) effects. Here on the contrary in the gravitational sector a Lorentz breaking is felt also locally, and for  $\tilde{S} \neq 0$  also rotations are broken. The physical effect is that gravitons will propagate differently in direction of the source.

We strongly believe that this breaking of (local) boosts *and* rotations at finite distance from a source is a general feature of nontrivial solutions in massive gravity, due to the presence of additional fields that can not be ‘gauged away’.

## 4. Conclusions

In this paper we approached the problem of finding a consistent massive deformation of gravity by introducing an additional spin 2 field  $g_2$  coupling non-derivatively to the standard metric field. This allows us to explore both the Lorentz invariant (LI) and Lorentz breaking (LB) phases working with consistent and dynamically determined backgrounds. Preserving diffeomorphisms and breaking Lorentz is also important; at the linearized level it forces  $\mathcal{M}_1$  to vanish and no dangerous scalar mode is propagating. Still at the linearized level, it was shown [8] that in the case  $\mathcal{M}_1 = 0$  the vDVZ discontinuity is absent, but a new linearly growing term is present in the static gravitational potential [10, 25], that seems to invalidate perturbation theory beyond some distance scale. To address this and the vDVZ discontinuity problem, we thus studied the exact spherically symmetric configurations. The exact solution that we found, valid for a large class of interaction potentials, shows that the linear term is replaced in the full solution by a power-like term  $r^\gamma$ , with  $\gamma$  depending on the nonlinear couplings in the interaction potential.

Phenomenologically, when  $\gamma < -1$  the total energy of the solution is finite and the space is asymptotically flat; Lorentz is broken in the gravitational sector by the asymptotic value of  $g_2$ , but normal matter only feels the modification of the gravitational potential. Using the full solution one can check that the absence of the vDVZ discontinuity is an exact result. The effect of the interaction manifests in the  $r^\gamma$  term whose size  $S$  was determined for a star by matching the exterior solution with an interior one. In addition to this, by the presence of the additional spin 2 field, the total mass of the star appearing in the Newton term gets a finite renormalization that depends on the object size, and may screen or even antiscreen the star mass. We believe that this is a general feature of massive gravity.

When  $g_1$  describes a black hole the solution depends not only on the collapsed mass but also on an other constant, probably remnant of the original shape; notice that there is no contradiction with the no-hair theorem because the Einstein equations are modified by the presence of  $Q_{1/2}$ .

In the case  $\gamma > -1$  the total energy is infinite and this may indicate only that solutions will not be spherically symmetric. For example the solution may be unstable under axially symmetric perturbations and drop to a flux tube in a sort of mass confinement scenario.

Indeed, regarding stability, even for the physical case  $\gamma < -1$  the final word would be given by studying the small fluctuations also around the exact solution, to check that the non-propagation of the (ghost) scalars and vectors is preserved on a nontrivial background. To this aim, we have discussed how the spontaneous Lorentz-breaking is present also in the nontrivial background, where we note that in general also rotations are locally broken. One expects this also to be a generic feature of massive gravity.

We showed that we can reach the LI phase by tuning  $c^2 \rightarrow 1$  in the exact solutions, and this results into a well behaved phase, though not the Fierz-Pauli one (gravitons are massless). The fate of the discontinuity and Vainshtein claim for the PF case is thus still an open problem and exact solutions of type II with  $c = 1$  are presently under investigation. Finally, it would be interesting to speculate on the role of the mass screening in cosmology, that may change the form of the Hubble expansion.

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## A. Background for solvable potentials

For the solvable potential (3.8), we report here the biflat solution as it results from solving the EOM (2.12), as well as the  $\mu^2$  and  $\lambda_2$  constants of the linearized analysis.

The fine tuning conditions to ensure flatness  $\mathcal{K}_1 = \mathcal{K}_2 = 0$  turn in two relations for the two cosmological constants. Defining  $\tilde{a}_n = \omega^{-2n}a_n$ ,  $\tilde{b}_n = \omega^{2n}b_n$  and  $\alpha_n = (\tilde{a}_n - c^2\tilde{b}_n)(c^2 - 1)/c^2$ , and  $\beta_n = (\tilde{a}_n + c^2\tilde{b}_n)$ , we have:

$$\begin{aligned}
 3\mathcal{K}_1 &= -2\omega^{-2}\Lambda_1 + \frac{c^{-3/2}}{(c^2-1)\gamma} \left[ -8c^2(3c^2 + \gamma + 1)\alpha_2 - 12c^2((\gamma + 6)c^2 + 3\gamma + 2)\alpha_3 \right] \\
 &\quad + c^{-3/2} \left[ (6c^2 - 2)\beta_2 + 24(c^2 - 1)\beta_3 - \tilde{a}_0c^2 + 3\tilde{b}_4c^4 - 5\tilde{a}_4 \right] \\
 3\mathcal{K}_2 &= -2\omega^2\Lambda_2\kappa^{-1} + \frac{c^{-5/2}}{(c^2-1)\gamma\kappa} \left[ 8((\gamma + 1)c^2 + 3)\alpha_2c^2 + 12((3\gamma + 2)c^2 + \gamma + 6)\alpha_3c^2 \right] \\
 &\quad - c^{-5/2} \left[ 2(c^2 - 3)\beta_2 + 24(c^2 - 1)\beta_3 + \tilde{a}_0c^2 + 5\tilde{b}_4c^4 - 3\tilde{a}_4 \right] \tag{A.1}
 \end{aligned}$$

and we remind that one of these is a genuine fine tuning to achieve flatness, as is usual in General Relativity, while the other is a complicated equation that may be used to find  $\omega$ . We prefer thinking in reverse and consider  $\omega$  a free parameter determining the right  $\Lambda_2$ . Finally, the lorentz breaking speed of light turns out to be

$$c^2 = -\frac{\tilde{a}_1 + 4\tilde{a}_2 + 6\tilde{a}_3}{\tilde{b}_1 + 4\tilde{b}_2 + 6\tilde{b}_3}. \tag{A.2}$$

The relevant quantities entering in the linearized analysis are the graviton mass  $\lambda_2$  and the  $\mu^2$  parameter, that have the following expressions:

$$\begin{aligned}
 M_{pl}^2\mu^2 &= \frac{(\gamma - 2)^2}{32} \left\{ \tilde{a}_0 - \frac{3}{c^2(c^2 - 1)(\gamma - 2)\gamma} \left[ 5(c^2 - 1)(\gamma - 2)\gamma(\tilde{a}_4 + c^4\tilde{b}_4) \right. \right. \\
 &\quad \left. \left. + 6c^2(1 + c^2)(\gamma - 2)\gamma\beta_2 + 32c^2(1 + c^2)(\gamma - 2)\gamma\beta_3 \right. \right. \\
 &\quad \left. \left. - 4c^2(c^2 - 1) [(\gamma + 2)^2\alpha_2 + (12 + 4\gamma + 7\gamma^2)\alpha_3] \right] \right\} \tag{A.3}
 \end{aligned}$$

$$M_{pl}^2m_g^2 = \lambda_2 = G\frac{2(\gamma - 2)}{\gamma}(\alpha_2 + 3\alpha_3). \tag{A.4}$$

## B. Interior solution

For generality we report here the case of a star composed of two spherical regions filled with incompressible fluids of kind 1 and 2 extending from the origin to different radii  $R_1$  and  $R_2$ ; of constant densities  $\rho_{1,2} = \frac{M_{1,2}}{4/3\pi R_{1,2}^3}$  and negligible pressures. In general we have two scenarios, for the three different regions:

- (a) for  $R_2 < R_1$  we have  $0 < r < R_2$ ,  $R_2 < r < R_1$  or  $r > R_1$ ;  
 (b) for  $R_1 < R_2$  we have  $0 < r < R_1$ ,  $R_1 < r < R_2$  or  $r > R_2$ .

We give only the analytic results for  $J[r]$ , that is the gravitational potential in  $g_1$ :

- Starting from the exterior solutions  $r > R_{1,2}$ , we find a common value for the exterior potential  $J$  in both scenarios a) and b):

$$J(r) = 1 - \frac{2G}{r} \left[ M_1 + \frac{16\mu^2 (M_1 R_1^2 - \omega^2 \kappa^{-1} M_2 R_2^2) \sqrt{c} \omega^2}{5(\gamma - 2)(\gamma + 1)} \right] + r^\gamma \left[ \frac{96G\mu^2 (\omega^2 M_2 R_2^{1-\gamma} - \kappa M_1 R_1^{1-\gamma}) \sqrt{c} \omega^2}{(\gamma - 4)(\gamma - 2)(\gamma + 1)(2\gamma - 1)\kappa} \right]. \quad (\text{B.1})$$

- The intermediate solutions are different in the two scenarios:

for a) i.e.  $R_2 < r < R_1$

$$1 - \frac{3GM_1}{R_1} + \frac{32G\sqrt{c}\mu^2\omega^4 M_2 R_2^2}{5r(2-\gamma)(\gamma+1)\kappa} + \frac{Gr^2 M_1 \left(1 - \frac{16\sqrt{c}\mu^2\omega^2 R_1^2}{(2-\gamma)(\gamma+1)}\right)}{R_1^3} - \frac{48Gr^4 \sqrt{c}\mu^2\omega^2 M_1}{5(\gamma-4)(\gamma+3)R_1^3} + \frac{96G\sqrt{c}\mu^2\omega^2 M_1 R_1^\gamma r^{1-\gamma}}{(\gamma-2)(\gamma+1)(\gamma+3)(2\gamma-1)} + \frac{96G\sqrt{c}\mu^2\omega^4 M_2 R_2^{1-\gamma} r^\gamma}{(\gamma-4)(\gamma-2)(\gamma+1)(2\gamma-1)\kappa} \quad (\text{B.2})$$

for b) i.e.  $R_1 < r < R_2$

$$1 - \frac{2GM_1}{r} - \frac{32G\sqrt{c}\mu^2\omega^2 M_1 R_1^2}{5r(2-\gamma)(\gamma+1)} - \frac{16Gr^2 \sqrt{c}\mu^2\omega^4 M_2}{(\gamma-2)(\gamma+1)\kappa R_2} + \frac{48Gr^4 \sqrt{c}\mu^2\omega^4 M_2}{5(\gamma-4)(\gamma+3)\kappa R_2^3} - \frac{96G\sqrt{c}\mu^2\omega^4 M_2 R_2^\gamma r^{1-\gamma}}{(\gamma-2)(\gamma+1)(\gamma+3)(2\gamma-1)\kappa} - \frac{96G\sqrt{c}\mu^2\omega^2 M_1 R_1^{1-\gamma} r^\gamma}{(\gamma-4)(\gamma-2)(\gamma+1)(2\gamma-1)}. \quad (\text{B.3})$$

- The inner solutions  $r < R_{1,2}$  have again a common form in both scenarios a) and b):

$$1 - \frac{3GM_1}{R_1} + Gr^2 \left[ M_1 \left( \frac{16\sqrt{c}\mu^2\omega^2}{(\gamma-2)(\gamma+1)R_1} + \frac{1}{R_1^3} \right) - \frac{16\sqrt{c}\mu^2\omega^4 M_2}{(\gamma-2)(\gamma+1)\kappa R_2} \right] + \frac{48G\sqrt{c}\mu^2\omega^2 (\omega^2 M_2 R_1^3 - \kappa M_1 R_2^3) r^4}{5(\gamma-4)(\gamma+3)\kappa R_1^3 R_2^3} + \frac{96G\sqrt{c}\mu^2\omega^2 (\kappa M_1 R_1^\gamma - \omega^2 M_2 R_2^\gamma) r^{1-\gamma}}{(\gamma-2)(\gamma+1)(\gamma+3)(2\gamma-1)\kappa}. \quad (\text{B.4})$$

As one checks, the solution is regular at the origin.

## C. Energy integrals

In the presence of a time-like Killing vector in GR one can define the notion of total gravitational energy as a flux from an asymptotic 2-surface that involve only the gravitational field at large distance, far from the sources. Consider the following metric

$$ds^2 = -C(r) dt^2 + 2D(r) dr dt + A(r) dr^2 + B(r) d\Omega^2, \quad (\text{C.1})$$

with the time-like Killing vector  $K = \frac{\partial}{\partial t}$ . From the Killing equation we have

$$\nabla^\mu J_\mu = 0, \quad J_\mu = \square K_\mu. \quad (\text{C.2})$$

The Komar energy  $\mathcal{E}$  is defined by

$$\mathcal{E} = w \int_{t=t_1} \sqrt{h} n^\mu J_\mu, \quad (\text{C.3})$$

where  $h_{\mu\nu}$  is the induced metric in the hyper-surface  $t = \text{const.}$  with unit normal  $n^\mu$  and  $w$  is a normalization constant. According to Stokes theorem, given a 3-surface  $V$ , for any antisymmetric tensor  $F_{\mu\nu}$  we have

$$\int_V d^3x \sqrt{h} n^\mu \nabla^\nu F_{\mu\nu} = \int_{\partial V} d^2x \sqrt{\gamma} (n^\alpha v^\beta - n^\beta v^\alpha) F_{\alpha\beta}, \quad (\text{C.4})$$

where  $v^\alpha$  is the unit normal to  $\partial V$  and  $\gamma_{\alpha\beta}$  is the induced metric in  $\partial V$ . Then Stokes theorem gives

$$\mathcal{E} = -\frac{w}{2} \int_{t=t_1, r=r_1} \sqrt{\gamma} (n^\alpha v^\beta - v^\alpha n^\beta) \nabla_\alpha K_\beta. \quad (\text{C.5})$$

In general the Komar energy will depend on 2-surface that bounds the  $t = \text{const.}$  slice. Indeed, from Einstein equations it easy to show that the difference  $\Delta\mathcal{E}$  between the Komar energy computed with two different bounding 2-surfaces  $\Sigma_1$  and  $\Sigma_2$  is proportional to the integral of the Ricci tensor over the 3-volume bounded by  $\Sigma_1$  and  $\Sigma_2$ . As a result the Komar energy does not depend on  $\Sigma$  in a region where the Ricci tensor is vanishing. This is indeed the case in a region far from any source. Then

$$\mathcal{E} = -\frac{w}{2} \int_{t=\text{const.}, r \rightarrow \infty} \sqrt{\gamma} (n^\alpha v^\beta - v^\alpha n^\beta) \nabla_\alpha K_\beta; \quad (\text{C.6})$$

for the induced metric on the 3-surface  $t = \text{const.}$  and its normal  $n$  we get

$$\begin{aligned} dl^2 &= A(r) dr^2 + B(r) d\Omega^2; \\ n &= (AC + D^2)^{-1/2} \left( -A^{1/2} \frac{\partial}{\partial t} + DA^{-1/2} \frac{\partial}{\partial r} \right) \end{aligned} \quad (\text{C.7})$$

and for the induced metric on  $t, r = \text{const.}$  and its normal  $v$  ( $v$  is normalized with  $h$ )

$$ds^2 = B(r) d\Omega^2, \quad v = A^{1/2} \frac{\partial}{\partial r}. \quad (\text{C.8})$$

We have then

$$\mathcal{E} = - \lim_{r \rightarrow \infty} \frac{4\pi w C' B}{\sqrt{D^2 + AC}}. \quad (\text{C.9})$$

One can recover the same result using the language of differential forms. Introducing the 1-form  $J = -(\delta d + d\delta)\tilde{K}$  in terms of the 1-form  $\tilde{K}$  associated with the Killing vector  $K$ . From the Killing equation  $\delta\tilde{K} = 0$ , then

$$J = -(\delta d + d\delta)\tilde{K} \equiv \delta d\tilde{K} = *d*d\tilde{K} = \square K_\mu dx^\mu. \quad (\text{C.10})$$

Now

$$\int d * J = 0 \Rightarrow \int_{t=\text{const.}} * \tilde{J} \quad \text{is time-independent} \quad (\text{C.11})$$

and finally

$$\begin{aligned} \mathcal{E} &= -w \int_{t=t_1} * J = -w \int_{t=t_1} * \delta d \tilde{K} = w \int_{t=t_1} * * d * d \tilde{K} = w \int_{t=t_1} d * d \tilde{K} \\ &= w \int_{t=t_1, r=r_1} * d \tilde{K} = \int_{t=t_1, r=r_1} \sqrt{g} \frac{C'}{2} g^{\mu t} g^{\nu r} \epsilon_{\mu\nu\alpha\beta} dx^\alpha \wedge dx^\beta \\ &= -\frac{4\pi w BC'}{\sqrt{D^2 + AC}}. \end{aligned} \quad (\text{C.12})$$

## D. Simplest bigravity

Here we analyze the system in the simpler particular case when  $V$  is a function of  $q$  only:  $V = f(q)$ . The Bianchi identities (2.8) for  $Q_{1,2}$ , can be written as<sup>12</sup>

$$\partial_\mu V - [\partial_\nu \log q + 2 (\nabla_{1\nu} - \nabla_{2\nu})] (V' X)_\mu^\nu = 0 \quad (\text{D.1})$$

$$8 (\nabla_{1\nu} + \nabla_{2\nu}) (V' X)_\mu^\nu - V \partial_\mu \log q = 0 \quad (\text{D.2})$$

and because in the case at hand  $(V' X)_\mu^\nu = f' q \delta_\mu^\nu$ , the only non-trivial equation is

$$\left[ 16 \frac{d^2}{d(\log q)^2} f - f \right] \partial_\mu q = 0. \quad (\text{D.3})$$

Thus, either  $q = \text{const.}$  or  $V = V_0 = c_1 q^{1/4} + c_2 q^{-1/4}$ . However,  $(g_1 g_2)^{1/4} V_0 = c_1 \sqrt{g_1} + c_2 \sqrt{g_2}$  would imply that the two sectors do not see each other and we are left with two independent copies of GR + cosmological term. The theory with  $q = \text{const.}$  is the simplest of all possible bigravity theories, the EOM reduce to

$$M_{pl1}^2 E_{1\nu}^\mu + \mathcal{K}_1 \delta_\nu^\mu = \frac{1}{2} T_{1\nu}^\mu \quad (\text{D.4})$$

$$M_{pl2}^2 E_{2\nu}^\mu + \mathcal{K}_2 \delta_\nu^\mu = \frac{1}{2} T_{2\nu}^\mu, \quad (\text{D.5})$$

with

$$\mathcal{K}_1 = q^{1/4} [f(q) - 4q f'(q)] \quad (\text{D.6})$$

$$\mathcal{K}_2 = q^{-1/4} [f(q) + 4q f'(q)]. \quad (\text{D.7})$$

The effective cosmological constants are thus related; moreover, this simplest bigravity, due to the constraint  $q = \text{const.}$ , is equivalent to a single GR + unimodular GR, and the two sectors share the conformal mode. Finally, besides the diagonal diff also two independent volume-preserving diffs are present.

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<sup>12</sup>And recall that for any vector field  $v_\mu$  one has  $(\nabla_{2\mu} - \nabla_{1\mu})v_\nu = C_{\mu\nu}^\sigma v_\sigma$ , with the tensor  $C_{\mu\nu}^\sigma = g_2^{\sigma\beta} (\nabla_{1\mu} g_{2\nu\beta} + \nabla_{1\nu} g_{2\mu\beta} - \nabla_{1\beta} g_{2\mu\nu}) / 2$ .

As an example of exact solution, we present the solution for the potential

$$V = \text{tr} \ln X = \ln \det X \tag{D.8}$$

(and we may add also the two cosmological constant terms  $\Lambda_1 q^{-1/4}$  and  $\Lambda_2 q^{1/4}$  to achieve flatness). With this potential we have  $V'X = \mathbf{1}$  by construction. The solution in general is Schwarzschild-deSitter for both metrics, but  $g_2$  is in a different gauge:

$$J = \Delta_1 \left( 1 - 2\frac{m_1}{r} + c_1 r^2 \right), \quad K = \Delta_1 / J, \tag{D.9}$$

$$C = \Delta_2 \left( 1 - 2\frac{m_2}{\rho} + c_2 \rho^2 \right), \quad \rho = (r^3 + \lambda^3)^{1/3} \tag{D.10}$$

$$B = \omega^2 \rho^2, \quad D^2 + AC = \Delta_2 \frac{(B')^2}{B} = c^2 \omega^4 \Delta_1 (\rho')^2, \tag{D.11}$$

$$A = \text{free}, \quad c^2 = \frac{4\Delta_2}{\omega^2 \Delta_1}. \tag{D.12}$$

This is a family of solutions because  $A(r)$  is a free function (!), remnant of the spatial diffs. The determinant  $AC + D^2$  is fixed by  $B(r)$ , and for  $\lambda \neq 0$  it is not constant, at finite distance. Then one can also use  $A$  to set  $D = 0$  and get a bidiagonal solution like (3.5). Notice that  $\omega^2$  and  $\Delta_2/\Delta_1$  are free constants, and so also the relative speed of light  $c^2$  is free.

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